MIXED BOUNDARY-VALUE PROBLEM FOR THE LAPLACE EQUATION IN THIN REGIONS OF SPECIAL SHAPE

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The method of asymptotic integration for elliptic equations in thin regions is extended to the case of boundary conditions of the third kind. Two steady-state heat-conduction problems in cylindrical objects are treated as examples.

So-called one-dimensional theories are frequently used in engineering calculations for complicated thermal, hydraulic, or magnetic circuits; for the stress states of structures, etc. For example, there is a one-dimensional equation for thin rods of high thermal conductivity; one-dimensional equations for the bending of thin rods, which are the limiting cases of the equations of the theory of elasticity; equations for long lines; etc. If the one-dimensional approximation turns out to be insufficiently accurate, a multidimen-problem is formulated, which usually does not make use of the information obtained in the one-dimensional approximation. There are many cases of practical importance in which small corrections of the one-dimensional theory are adequate, and it is not necessary to resort to a solution of the complete problem, which, generally speaking, can be found only for objects of extremely simple geometry with boundary conditions. The existing versions of the one-dimensional theories are usually not adequate for finding these corrections, which are "irrational" approximations, in the terminology of [1]. Recent decades have seen the development of so-called special-perturbation methods, which permit the irrational approximations to be converted into rational approximations; i.e., they permit a systematic determination of the corrections of all higher orders to the simple limiting solution. Many examples of one of these methods — the method of composite asymptotic expansions — are given in [1, 12]. A rigorous derivation for a similar method, called the "method of boundary-layer corrections" in [8], is given in [2-4]. An asymptotic integration of the Laplace equation was used in [12] to derive a one-dimensional heat-conduction equation for a thin rod with a thermally insulated lateral surface. It has been shown [5] that the method of asymptotic integration can also be applied to problems with boundary conditions of the third kind, if the coefficient of the unknown function is sufficiently small. Among such problems are those of the theory of heat conduction with a slight convective heat transfer at the boundary, the problem of calculating a magnetic circuit in the case in which there are narrow air-filled gaps [13], and several others.

§1. As an example we consider the steady-state temperature distribution in a thin cylinder at whose lateral surface there is a slight convective heat transfer with piecewise-constant heat-transfer coefficient. We choose the boundary conditions at the ends in the simplest form in order to concentrate on those features of the problem which are associated with the boundary condition of the third kind at the lateral surface. The problem reduces to seeking a function which satisfies the Laplace equation

$$\Delta\bar{u}(\bar{r},\ \bar{z})=0 \quad (0<\bar{r}<a,\quad 0<\bar{z}<l) \tag{1.1}$$

and the boundary conditions

$$\bar{u}(\bar{r},\ 0)=T_0,\ \bar{u}'_{\bar{z}}(\bar{r},\ l)=0,\ \bar{u}'_{\bar{r}}(a,\ \bar{z})=-h(\bar{z})\,\bar{u}(a,\ \bar{z}),$$
$$|\bar{u}(0,\ \bar{z})|<\infty, \tag{1.2}$$

where

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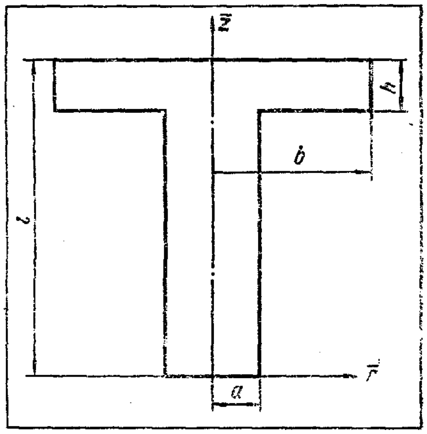

Fig. 1. Thin rod with a disk-shaped fin.

$$h(\bar{z}) = \begin{cases} h_1, & 0 < \bar{z} < l_1 \\ h_2, & l_1 < \bar{z} < l \end{cases} \text{ and } h_1,\ h_2,\ T_0 = \text{const.}$$

We seek an asymptotic expansion of the solution of problem (1.1), (1.2) for the case of a thin cylinder, i.e., for small $\varepsilon = a/l$. We also assume that the heat-transfer coefficient is small; more precisely, we assume

$$h_i = \frac{a}{l^2} H_i,\ H_i = O(1)\quad (\varepsilon \to 0),\ i = 1,2. \tag{1.3}$$

If the heat transfer is more pronounced, it is not possible to obtain a one-dimensional theory as a zeroth approximation, and the asymptotic behavior of the solution of mixed problem (1.1), (1.2) is constructed by analogy with the asymptotic behavior of the first boundary-value problem [7].

We introduce dimensionless variables, setting

$$\bar{u} = T_0 u,\ \bar{r} = ar,\ \bar{z} = lz,\ l_1 = lL_1. \tag{1.4}$$

The dimensionless heat-transfer coefficient is introduced by Eq. (1.3). Near the discontinuity of the function H(z) we also introduce the small-scale coordinates $\zeta_1 = (z - L_1)/\varepsilon$; $\rho_1 = r$; and near the lower end of the cylinder we introduce $\zeta_2 = z/\varepsilon$, $\rho_2 = r$. Following the method of boundary-layer corrections we seek a solution in the form

$$\bar{u}(\bar{r}, \bar{z}) = w(r, z) + v^{(1)}(\rho_1, \zeta_1) + v^{(2)}(\rho_2, \zeta_2);$$

a separate iterative process is worked out for each of the functions w and $v^{(i)}$'$(i = 1, 2)$.

In terms of the variables in (1.4) the problem of finding the function w, which depends on the large-scale coordinates, becomes

$$\frac{1}{r}(rw'_r)'_r + \varepsilon^2 w''_{zz} = 0 \quad (0 < r < 1,\ 0 < z < 1); \tag{1.1a}$$

$$w(r, 0) = 1,\ w'_z(r, 1) = 0,\ w'_r(1, z) = -\varepsilon^2 H(z)\, w(1, z),\ |w(0, z)| < \infty. \tag{1.2a}$$

We seek a solution of (1.1a), (1.2a) in the form

$$w = \sum_{k=0}^{\infty} \varepsilon^{2k} w_k. \tag{1.5}$$

Substituting (1.5) into (1.1a) and conditions (1.2a), and equating the coefficients of identical powers of ε, we find the following chain of problems for seeking the functions:

$$\frac{1}{r}\left(r\frac{\partial}{\partial r} w_k\right)'_r + \frac{\partial^2}{\partial z^2} w_{k-1} = 0,\ \frac{\partial}{\partial r} w_k(1, z) + H(z)\, w_{k-1}(1, z) = 0,\ |w_k(0, z)| < \infty. \tag{1.6}$$

For brevity we stipulate that in Eqs. (1.6) and everywhere below quantities with negative indices are assumed equal to zero. Setting $k = 0$ in (1.6) we find $w_0 = \bar{w}_0(z)$, where $\bar{w}_0(z)$ is an arbitrary function, to be determined. The problem for w_1 which is found from (1.6) with $k = 1$ is a one-dimensional second boundary-value problem, and the condition for the solvability of this problem is an equation for the function $\bar{w}_0$:

$$z[\bar{w}_0] \equiv \bar{w}''_0 - 2H(z)\bar{w}_0 = 0. \tag{1.7}$$

If (1.7) holds we can find a function w_1 in the form

$$w_1 = -\frac{1}{2} r^2 H(z)\bar{w}_0 + \bar{w}_1(z), \tag{1.8}$$

where $\bar{w}_1(z)$ is an arbitrary function, determined from the condition for the solvability of the problem for w_2.

We note that Eq. (1.7) is a particular case of the familiar equation of one-dimensional theory for a thin rod [6]; in terms of dimensional variables this equation is

$$\bar{w}''_0 - (hP/S)\,\bar{w}_0 = 0, \tag{1.9}$$

where P is the perimeter and S is the cross-sectional area of the rod.

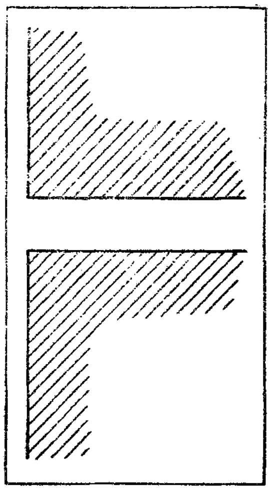

Fig. 2

It is natural to impose the first two of the boundary conditions in (1.2a) on the function $\bar{w}_0$. The discrepancy in the boundary condition at $z=0$ for the function w_1 given by Eq. (1.8) is balanced by the function $v^{(2)}$, whose local nature imposes the condition $v^{(2)} \to 0$ in the limit $\zeta_1 \to \infty$. If we seek the function $v^{(2)}$ in the form $(v^2) = \varepsilon^2 v_0^{(2)} + \varepsilon^3 v_1^{(2)} + \ldots$, the zeroth approximation $v_0^{(2)}$ turns out to be the solution of the boundary-value problem

$$\Delta v_0^{(2)}(\rho_2, \zeta_2) = 0 \quad (0 < \rho_2 < 1,\ 0 < \zeta_2 < \infty), \tag{1.10}$$

$$\frac{\partial}{\partial \rho_2} v_0^{(2)}(1, \zeta_2) = 0, \quad |v_0^{(2)}(0, \zeta_2)| < \infty,$$

$$v_0^{(2)}(\rho_2, 0) = \varphi(\rho_2) = \frac{1}{2} A\rho_2^2 - q_1, \tag{1.11}$$

where q_1 determines the condition at $z=0$ for the function $\bar{w}_1$. In this case the solution for $v_0^{(2)}$ is

$$v_0^{(2)} = \sum_{k=1}^{\infty} B_k J_0(\alpha_k \rho_2) \exp(-\alpha_k \zeta_2), \tag{1.12}$$

where α_k are the positive roots of the equation $J_1(\alpha) = 0$. Since this expansion does not contain a term corresponding to an eigenvalue of zero, it is clear that the function $\varphi(\rho_2)$ must be orthogonal to unity; we thus have the explicit form of the condition on $\bar{w}_1$ at $\zeta_2 = 0$. By expanding $\varphi(\rho_2)$ in terms of the orthogonal system of eigenfunctions we can find the expansion coefficients in (1.12) in the form $B_k = 2A/\alpha_k^2 J_0(\alpha_k)$. If r-dependent functions appeared in the conditions at the ends, the boundary layers near the points $z = 0$, 1 would have appeared in lower-order approximations.

Pursuing the first iterative process, we find, in each step, a function w_k with an accuracy to within some unknown term $\bar{w}_k(z)$, determined from the condition for the solvability of the problem for w_{k+1}. This condition takes the form of the homogeneous equation $z[\bar{w}_k] = f_k(z)$, where f_k is governed by the preceding approximations. Here the conditions for $\bar{w}_k(0)$ $(k = 1, 2, \ldots)$ are determined from the corresponding problems for the functions $v_{k-1}^{(2)}$. Furthermore, the functions $\bar{w}_k$ must satisfy some special conditions at $z = L_1$. As we see from (1.8), the discontinuity of the heat-transfer coefficient $H(z)$ leads to the appearance of discontinuities in the function w_1 and in its derivatives. For the same reason, discontinuities appear in all subsequent w_k. In a certain average sense we can compensate for these discontinuities by specifying the conditions

$$[\bar{w}_k(L_1)] = M_k, \quad [\bar{w}_k'(L_1)] = K_k,$$

where $[\psi(z)] = \psi(z+0) - \psi(z-0)$. Complete compensation of the discontinuities can be achieved only by means of the function $v^{(1)}$; the values of the constants M_k and K_k are determined in the course of the third iterative procedure.

We turn now to the determination of the function $v^{(1)}(\rho_1, \zeta_1)$. In terms of the variables (ρ_1, ζ_1), the coordinates of the ends are $(1 - L_1)/\varepsilon$ and $-L_1/\varepsilon$, respectively. Seeking the asymptotic behavior of the solution for small ε, we should assume these coordinates to be $\pm\infty$, neglecting the error, which is of the order of $\exp(-1/\varepsilon)$. The local nature of the function $v^{(1)}$ dictates the conditions $v^{(1)} \to 0$ at $\zeta_1 \to \pm\infty$ for it. With $\rho_1 = 0$ or $\rho_1 = 1$, the function $v^{(1)}$ satisfies the same conditions as the function w, i.e., conditions (1.2a). In the case $\zeta_1 = 0$ we must specify the discontinuities in $v^{(1)}$ and $v_{\zeta_1}^{(1)'}$ such that the sum $w + v^{(1)}$ is continuous along with its first derivative with respect to z:

$$[v^{(1)}(0)] = [w(L_1)] = \sum_{k=0}^{\infty} \varepsilon^{2k} [w_k(L_1)],$$

$$[v_{\zeta_1}^{(1)'}(0)] = \varepsilon [w_z'(L_1)] = \sum_{k=0}^{\infty} \varepsilon^{2k+1} \left[\frac{\partial}{\partial z} w_k(L_1)\right].$$

Using these latter equations, we seek the function $v^{(1)}$, which is sinusoidal for $\zeta_1 \gtrless 0$, in the form $v^{(1)} = \varepsilon^2 v_0^{(1)} + \varepsilon^3 v_1^{(1)} + \ldots$. Equating the coefficients of identical powers of ε in the Laplace equation and the boundary conditions of the type in (1.20), we find a chain of problems for finding the function:

$$\Delta v_n^{(1)}(\rho_1, \zeta_1) = 0; \quad v_n^{(1)} \to 0, \quad \zeta_1 \to \pm\infty; \quad |v_n^{(1)}(0, \zeta_1)| < \infty;$$

$$\frac{\partial}{\partial \rho_1} v_n^{(1)}(1, \zeta_1) = -H(\zeta_1)\, v_{n-2}^{(1)} \quad (n = 0, 1, 2, \ldots); \tag{1.13}$$

$$[v_{2k}^{(1)}(0)] = [w_{k+1}(L_1)], \quad \left[\frac{\partial}{\partial \zeta_1} v_{2k}^{(1)}(0)\right] = [v_{2k+1}^{(1)}(0)] = 0,$$

$$\left[\frac{\partial}{\partial \zeta_1} v_{2k+1}^{(1)}(0)\right] = \left[\frac{\partial}{\partial z} w_{2k+1}(L_1)\right].$$

To now determine the functions $v_0^{(1)}$ and $v_1^{(1)}$, we note that, according to (1.13), we have

$$[v_0^{(1)}(0)] = \frac{1}{2}\rho_1^2\, [H_1 \bar{w}_0 (L_1 - 0) - H_2 \bar{w}_0 (L_1 + 0)] - M_1 = \mu_0 \rho_1^2 - N_0,$$

$$\left[\frac{\partial}{\partial \zeta_1} v_1^{(1)}(0)\right] = \frac{1}{2}\rho_1^2\, [H_1 \bar{w}_0' (L_1 - 0) - H_2 \bar{w}_0' (L_1 + 0)] - K_1 = \mu_1 \rho_1^2 - N_1. \tag{1.14}$$

The functions $v_0^{(1)}$ and $v_1^{(0)}$ are found by the method of separation of variables:

$$v_i^{(0)} = C_{0,\, i} + \sum_{k=1}^{\infty} C_{k,\, i} J_0(\gamma_k \rho_1) \exp(-\gamma_k \zeta_1), \quad \zeta_1 > 0,$$

$$(i = 0, 1), \tag{1.15}$$

$$v_i^{(1)} = D_{0,\, i} + \sum_{k=1}^{\infty} D_{k,\, i} J_0(\gamma_k \rho_1) \exp(+\gamma_k \zeta_1), \quad \zeta_1 < 0,$$

where γ_k are the positive roots of the function $J_1(\gamma)$. Writing the right sides of (1.14) in the form

$$\mu_i \rho_1^2 - N_i = \frac{1}{2}\mu_i + 4\mu_i \sum_{k=1}^{\infty} \frac{J_0(\gamma_k \rho_1)}{\gamma_k^2 J_0(\gamma_k)} - N_i \quad (i = 0, 1) \tag{1.16}$$

and substituting (1.15) and (1.16) into the boundary conditions, we find the following system of equations for the constants $C_{k,i}$ and $D_{k,i}$:

$$C_{0,0} - D_{0,0} = \frac{1}{2}\mu_0 - M_1; \quad C_{0,1} - D_{0,1} = 0;$$

$$C_{k,0} - D_{k,0} = 4\mu_0 / \gamma_k^2 J_0(\gamma_k);$$

$$C_{k,1} - D_{k,1} = 0; \quad C_{k,0} + D_{k,0} = 0; \tag{1.17}$$

$$C_{k,1} + D_{k,1} = 4\mu_1 / \gamma_k^2 J_0(\gamma_k).$$

The condition $v^{(1)} \to 0$ can be satisfied in the limits $\zeta_1 \to \pm\infty$ only if $M_1 = {}^1/_2\mu_0$, and the second condition in (1.14) can be satisfied only if $K_1 = {}^1/_2\mu_1$. These requirements determine the remaining unknown constants, M_1 and K_1. The other coefficients are found from (1.17):

$$C_{0,0} = D_{0,0} = C_{0,1} = D_{0,1} = 0; \quad C_{k,i} = (-1)^i D_{k,i} = 2\mu_i / \gamma_k^2 J_0(\gamma_k).$$

The succeeding functions $v_k^{(1)}$ are found analogously. The requirement that $v_k^{(1)}$ decay at infinity and the requirement of solvability of the equations $[(\partial/\partial\zeta_1) v_{2k+1}^{(1)}(0)] = [(\partial/\partial z) w_{k+1}(L_1)]$ lead to the condition that the discontinuities of $\bar{w}_k$ and $\bar{w}_k'$ are orthogonal, with a weight r, to the identity element; hence we can determine M_k and K_k. Knowing M_k and K_k, we can completely determine the function $\bar{w}_k$ and continue the iterative process as far as we wish.

We now consider the case in which the heat transfer at part of the lateral surface is more intense than was assumed above. For example, we assume that Eq. (1.3), which can be written in the form $\mathrm{Bi} = O(\varepsilon^2)$, where we have used the dimensionless Biot number, holds only for $z < L_1$, while for $z > L_1$ we have $\mathrm{Bi} = O(1)$. Then repeating the procedure above, we can show that we have $w \equiv 0$ for $z > L_1$. The function $v^{(1)}$ at $\zeta_1 < 0$ is given by (1.15), while for $\zeta_1 > 0$ we must seek the function $v_k^{(1)}$ in the form

$$v_k^{(1)} = \sum_{i=1}^{\infty} C_i J_0[\gamma_i^{(1)} \rho_1] \exp[-\gamma_i^{(1)} \zeta_1], \tag{1.18}$$

where $\gamma_i^{(1)}$ are the positive roots of the equation $\mathrm{Bi}\, J_0(\gamma) = \gamma J_1(\gamma)$. Satisfying the boundary conditions at $\zeta_1 = 0$, we find a completely regular infinite system of linear algebraic equations for the expansion coefficients $v_0^{(1)}$ for $\zeta_1 \gtrless 0$. The solution of this system of equations can be found easily by the reduction method.

§2. We now consider the problem for an object of a more complicated shape; specifically, we consider the problem of the theory of heat conduction for a thin rod with a disk-shaped fin (Fig. 1). We assume that the entire surface of the object except the circle $\bar{z}=l$ is thermally insulated, and we assume that there is convective heat transfer at the upper surface of the disk:

$$-\lambda u'_{\bar{z}}(\bar{r}, l)=\alpha u(\bar{r}, l). \quad (2.1)$$

We assume the thickness of this disk to be a quantity of the same order of smallness as the rod radius; i.e., we assume $a/l=\varepsilon$, $h/b=H\varepsilon$, $H=O(1)$, and $b/l=O(1)$ in the limit $\varepsilon\to 0$. In terms of the dimensionless variables (r, z) the problem can be stated as follows: We are to find the function $u(r, z)$ which satisfies the equation

$$\frac{1}{r}(ru'_r)'_r+\varepsilon^2 u''_{zz}=0 \quad (2.2)$$

in the region shown by Fig. 1 with the boundary conditions

$$u(r, 0)=T_0,\ |u(0, z)|<\infty,\ u'_z(r, 1)=-A\varepsilon u(r, 1),\ A=\frac{\alpha l^2}{\lambda a}; \quad (2.3)$$

over the rest of the surface the normal derivative of the function u vanishes.

In terms of the variables (r, z) it is convenient to seek a solution in the cylindrical part of the region, $0<r<1$, $0<z<1$. To construct the solution in the disk, $a<\bar{r}<b$, $l-h<\bar{z}<l$, it is preferable to choose the coordinates $Z=(l-\bar{z})/h$, $R=r/b$. The boundary layers formed at the intersection of the disk and the cylinder, as before, are constructed in terms of the small-scale variables $\rho_1=\bar{r}/a$, $\zeta_1=(l-\bar{z})/a$, while the boundary layer at the $\bar{z}=0$ end is constructed of the variables $\rho_2=\bar{r}/a$, $\zeta_2=\bar{z}/a$. We seek an approximate solution of the problem in the form

$$u=u^{(1)}(r, z)+u^{(2)}(R, Z)+v^{(1)}(\rho_1, \zeta_1)+v^{(2)}(\rho_1, \zeta_1)+v^{(3)}(\rho_2, \zeta_2).$$

Here the function $u^{(1)}$ is determined in the cylinder $0<r<1$, $0<z<1$; the function $u^{(2)}$ is determined in the region $\varepsilon/B<R<1$ (where $B=b/l$), $0<Z<1$; and the boundary layers $v^{(1)}$, $v^{(2)}$, and $v^{(3)}$ are determined in the regions $0<\rho_1<1$, $0<\zeta_1<\infty$; $1<\rho_1<\infty$, $0<\zeta_1<BH$; and $0<\rho_2<1$, $0<\zeta_2<\infty$. Equations for the functions $u^{(i)}$ $(i=1, 2)$ and $v^{(k)}$ $(k=1, 2, 3)$ and the conditions at the surface of the cylinder and the disk are found from (2.2)-(2.3) after the appropriate change of variables. The requirements that the function and its normal derivative be continuous at the disk–cylinder interface lead to the relations

$$v^{(1)}(1, \zeta_1)-v^{(2)}(1, \zeta_1)=u^{(2)}\left(\frac{\varepsilon}{B}, Z\right)-u^{(1)}(1, z), \quad (2.4)$$

$$v^{(1)\prime}_{\rho_1}(1, \zeta_1)-v^{(2)\prime}_{\rho_1}(1, \zeta_1)=\frac{\varepsilon}{B}u^{(2)\prime}_R\left(\frac{\varepsilon}{B}, Z\right)-u^{(1)\prime}_r(1, z) \quad (2.5)$$

$$0<\zeta_1<BH,\ 0<Z<1,\ 1-h/l<z<1.$$

The sharp break at the boundary of the object leads to the appearance of logarithmic terms in the asymptotic expansion of the solution in ε^0, as in the hydrodynamic problem of flow around a rectangular profile [1].

The function $v^{(3)}(\rho_2, \zeta_2)$ is constructed in a manner similar to that used above. Then, using the same arguments, we can see that the term of the order of ε^0 in the expansion of $u^{(1)}$ depends only on z and satisfies an equation like (1.7) with $h=0$. Analogously, we can obtain an equation for the first nonvanishing term in the expansion of the function $u^{(2)}$: the term $u_0^{(2)}=\bar{u}_0^{(2)}$ to that (R). This term is given in order of magnitude by

$$(R\bar{u}_0^{(2)\prime})'-(AB/H)R\bar{u}_0^{(2)}=0. \quad (2.6)$$

The solution of (2.6) which satisfies the boundary condition of the second kind at $R=1$ is

$$\bar{u}_0^{(2)}(R)=C_0^{(2)}\left[I_0\left(\sqrt{\frac{AB}{H}}R\right)K_1\left(\sqrt{\frac{AB}{H}}\right)+I_1\left(\sqrt{\frac{AB}{H}}\right)K_0\left(\sqrt{\frac{AB}{H}}R\right)\right]. \quad (2.7)$$

Knowing the behavior of the modified Bessel functions for small arguments, we can find the asymptotic values of $u_0^{(2)}$ at the disk–cylinder interface:

$$\bar{u}_0^{(2)}\left(\frac{\varepsilon}{B}\right)=-C_0^{(2)}I_1\left(\sqrt{\frac{AB}{H}}\right)\ln\varepsilon+O(1),\ \bar{u}_0^{(2)\prime}\left(\frac{\varepsilon}{B}\right)=-BC_0^{(2)}I_1\left(\sqrt{\frac{AB}{H}}\right)\frac{1}{\varepsilon}+O(\varepsilon\ln\varepsilon). \quad (2.8)$$

Equations (2.8) show that if the solution is to remain bounded as $\varepsilon \to 0$ there must be no term of the order of ε^0 in the expansion for $u^{(2)}$. Here the functions $u^{(1)}$ and $u^{(2)}$ can "grow together" only if the condition $\bar{u}_0^{(1)}(1)=0$ holds. The function $u_0^{(1)}$ determined by this condition cannot satisfy the third of the conditions in (2.3). The resulting discrepancy, of the order of ε, must be balanced, according to the condition

$$\varepsilon u_z^{(1)'} + v_{\zeta_1}^{(1)'} = -A\varepsilon^2 [u^{(1)} + v^{(1)}], \ \zeta_1 = 0\,(z=0), \ 0 < r, \ \rho_1 < 1, \tag{2.9}$$

[which follows from (2.3)], by the function $v^{(1)}$. The expansion of this function, like that for $v^{(2)}$, must begin with the term $\varepsilon v_0^{(1)}$. The continuity of the heat flux from the rod into the disk [Eq. (2.5)], along with the asymptotic equations in (2.8), dictates the choice of power of ε in the expansion of $u^{(2)}$, i.e., $u^{(2)} = \varepsilon \bar{u}_0^{(2)} + \ldots$. Finally, the first equation in (2.8), along with (2.4), implies the presence of logarithmic terms in the expansion of $u^{(1)}$, namely, $u^{(1)} = \bar{u}_0^{(1)} + \varepsilon \bar{u}_1^{(1)} + \varepsilon \ln \varepsilon u_1{}^{(1)}{}_1 + \ldots$.

Using the terms of the expansions written out above and the conditions for the matching of the large-scale functions $u^{(1)}$ and $u^{(2)}$ and the functions of the boundary-layer type, $v^{(1)}$ and $v^{(2)}$ [Eqs. (2.4), (2.5), and (2.9)], we can write the final formulation of the problem for $v_0^{(1)}$ and $v_0^{(2)}$: We are to find functions which are sinusoidal in the regions $0 < \rho_1 < 1$, $0 < \zeta_1 < \infty$ and $1 < \rho_1 < \infty$, $0 < \zeta_1 < BH$ and which satisfy the conditions

$$\begin{gathered} \frac{\partial}{\partial \zeta_1} v_0^{(1)}(\rho_1, 0) = -T_0; \ \frac{\partial}{\partial \rho_1} v_0^{(1)}(\rho_1, BH) = 0; \\ \frac{\partial}{\partial \zeta_1} v_0^{(2)}(\rho_1, BH) = \frac{\partial}{\partial \zeta_1} v_0^{(2)}(\rho_1, 0) = 0, \\ v_0^{(1)} \to 0 \ (\zeta_1 \to \infty), \quad v_0^{(2)} \to 0 \ (\rho_1 \to \infty) \end{gathered} \tag{2.10}$$

and the joining conditions

$$\begin{gathered} \frac{\partial}{\partial \rho_1} v_0^{(1)}(1, \zeta_1) = q + \frac{\partial}{\partial \rho_1} v_0^{(2)}(1, \zeta_1), \\ v_0^{(2)}(1, \zeta_1) = v_0^{(1)}(1, \zeta_1) - T_0\zeta_1 + A_2. \end{gathered}$$

The quantity q, which is related to the one-dimensional solution in the disk, $\bar{u}_0^{(2)}$, is to be determined from the condition for solvability of the problem of $v_0^{(1)}$, since for arbitrary $C_0^{(2)}$ and A_2 it is not possible to satisfy the decay conditions in the limits $\rho_1 \to \infty$ and $\zeta_1 \to \infty$. Using the method of separation of variables, we can write $v_0^{(1)}$ and $v_0^{(2)}$ as

$$v_0^{(1)} = \sum_{k=1}^{\infty} D_k \exp(-\mu_k \zeta_1) J_0(\mu_k \rho_1) \ (0 < \rho_1 < 1, \ BH < \zeta_1 < \infty);$$

$$v_0^{(1)} = \frac{T_0}{2BH}\left(\zeta_1^2 - 2BH\zeta_1 - \frac{\rho_1^2}{2}\right) + C_0 - \sum_{k=1}^{\infty} D_k \exp(-\mu_k BH)$$

$$\times \frac{\operatorname{ch}\mu_k\zeta_1}{\operatorname{sh}\mu_k BH} J_0(\mu_k\rho_1) - \sum_{k=1}^{\infty} E_k \frac{K_1(k\pi/BH)}{I_1(k\pi/BH)} I_0(k\pi\rho_1/BH) \cos(k\pi\zeta_1/BH) \ (0 < \rho_1 < 1, \ 0 < \zeta_1 < BH);$$

$$v_0^{(2)} = \sum_{k=1}^{\infty} E_k \cos(k\pi\zeta_1/BH) K_0(k\pi\rho_1/BH) \ (1 < \rho_1 < \infty, \ 0 < \zeta_1 < BH), \tag{2.11}$$

where μ_k are the positive roots of the equation $J_1(\mu) = 0$.

The condition for the solvability of the problem for the region $(0 < \zeta_1 < BH, \ 0 < \rho_1 < 1)$ leads to an explicit expression for the quantity $q = -T_0/2BH$. Since the first and third expansions in (2.11) do not contain terms with eigenfunctions corresponding to the eigenvalue of zero, the orthogonality conditions

$$\int_0^1 v_0^{(1)}(\rho_1, BH)\rho_1 d\rho_1 = 0$$

and

$$\int_0^{BH} v_0^{(2)}(1, \zeta_1)\, d\zeta_1 = 0 \tag{2.12}$$

must hold. From these conditions we can find explicit expressions for C_0 and A_2:

$$C_0 = \frac{T_0 BH}{2} + \frac{T_0}{8BH} + 2\sum_{k=1}^{\infty} E_k (-1)^k \frac{BH}{k\pi} K_1\left(\frac{k\pi}{BH}\right),$$

$$A_2 = -\frac{2T_0 BH}{3} + \frac{T_0}{8BH} + \frac{1}{BH}\sum_{k=1}^{\infty} D_k \exp(-\mu_k BH)\frac{J_0(\mu_k)}{\mu_k} - 2\sum_{k=1}^{\infty} E_k (-1)^k \frac{BH}{k\pi} K_1\left(\frac{k\pi}{BH}\right). \quad (2.13)$$

Using the conjugate conditions (2.10) to determine the coefficients D_k and E_k in expansions (2.11), we can find a set of two infinite systems of algebraic equations. Using methods analogous to those in [10], we can show this set of two systems is completely regular; this result allows us to, in turn, find the unknowns D_k and E_k by the reduction method with any specified accuracy [11].

The substituting of the resulting values of these unknowns into (2.11) completely determines the first terms of the asymptotic expansions of the corresponding functions and permits us to begin a new cycle of iterations. The entire iterative process can be pursued as long as we wish; any term of the type $\varepsilon^p \ln^q \varepsilon u^{(1)}_{p,q}$ generates terms $\varepsilon^{p+1} \ln^q \varepsilon v^{(1)}_{p+1,q}$ and $\varepsilon^{p+2} \ln^q \varepsilon u^{(2)}_{p+2,q}$, which in turn generate terms of the type $\varepsilon^{p+2} \ln^{q+1} \times \varepsilon u^{(1)}_{p+2,q+1}$ and $\varepsilon^{p+1} \ln^q \varepsilon u^{(1)}_{p+1,q}$; i.e., there is a sort of chain reaction, which doubles the number of terms in the expansion after each cycle.

In conclusion, we wish to point out certain features of the iterative process for the case in which the disk is "thinner" than the rod, i.e., in the case $h/b = H_1\varepsilon^2$, $H_1 = O(1)$ in the limit $\varepsilon \to 0$. We assume that the coordinate of the upper surface of the disk is $\bar{z} = l_1$; in general, this surface may not coincide with the upper end of the cylinder. Introducing the coordinates $X = (\bar{r} - a)/h$, $Y = (\bar{z} - l_1)/h$, we find that the function $v^{(1)}$ satisfies

$$\frac{\partial}{\partial X}\left[(1+\varepsilon X)\frac{\partial v^{(1)}}{\partial X}\right] + (1+\varepsilon X)\frac{\partial^2 v^{(1)}}{\partial Y^2} = 0 \quad (2.14)$$

in the region shown in Fig. 2. Accordingly, each term in the expansion of $v^{(1)}$ satisfies a two-dimensional Laplace equation in Cartesian coordinates. The logarithmic terms in the asymptotic expansion appear because of the logarithmic singularity of the function performing the conformal mapping of the region (Fig. 2) into a band. The functions $u^{(1)}$ and $u^{(2)}$ "are interchanged" with the boundary conditions. The heat flux from the rod into the disk is determined in each approximation from the condition for the solvability of the one-dimensional problem in the disk. This heat flux determines the magnitude of the discontinuity of the derivative $u^{(1)}$ at the cross section of the disk for terms of higher order, and the terms in the expansion of $u^{(1)}$ determined in accordance with this condition give the value of the temperature in the disk.

NOTATION

u, dimensionless temperature; ε, small parameter of the problem; $\bar{r}$, $\bar{z}$, dimensional coordinates; r, z, R, Z, ρ, ζ, dimensionless variables; $Bi = \alpha a/\lambda$, Biot number; α, heat-transfer coefficient; λ, thermal conductivity; A, B, H, dimensionless coefficients; $J_0(x)$, $J_1(x)$, Bessel functions of the first kind; $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$, modified Bessel functions.

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